Some Remarks on a Weighted Least-Squares Finite Element Method for Second-Order Boundary Value Problems

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ABSTRACT

The mathematical analysis of the optimal order of convergence in $H^1$-norm for a weighted least-squares finite element method proposed by Chang is incorrect (Least-squares finite elements for second-order boundary value problems with optimal rates of convergence, Applied Mathematics and Computation, Vol. 76, pp. 267-284, 1996). In this note, using the technique of the Gauss projection with an additional inverse assumption, we prove that the assertion is still valid. Some remarks are also given.

(KEY WORDS: Boundary Value Problems; Finite Element Methods; Least-Squares; Convergence; Error estimates)

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I. INTRODUCTION

During the past twenty years, engineers and mathematicians have successfully developed the mixed finite element methods for solving the following first-order elliptic boundary value problem, which is reformulated from some second-order partial differential equation,

$$ A\nabla u - p = 0 \quad \text{in } \Omega, \quad (1) $$

$$ \nabla \cdot p + f = 0 \quad \text{in } \Omega, \quad (2) $$
where \( \Omega \subseteq \mathbb{R}^2 \) is a bounded domain with smooth boundary \( \Gamma := \partial \Omega \), \( u \) and \( p = (p_1, p_2)^T \) are the unknown functions, and the symmetric matrix

\[
A(x) = \begin{bmatrix}
a_{11}(x) & a_{12}(x) \\
a_{21}(x) & a_{22}(x)
\end{bmatrix}_{2 \times 2}
\]

defined on \( \overline{\Omega} \) is assumed to be smooth and satisfies the following uniform ellipticity condition:

\[
\xi^T A \xi \geq \alpha \| \xi \|^2 \quad \text{on} \quad \overline{\Omega},
\]

for all \( \xi = (\xi_1, \xi_2)^T \in \mathbb{R}^2 \), and \( \alpha \) is a positive constant independent of \( \xi \).

The mixed finite element formulation explicitly involves the flux \( p \) as a new dependent variable, hence accurate nodal fluxes can be obtained directly from the discretized mixed system. However, it leads to a saddle point problem in which the approximating function spaces have to satisfy the so-called inf-sup consistency condition of Ladyzhenskaya-Babuška-Brezzi [1]. Thus, many seemingly natural finite elements are never admissible.

In recent years there has been significant progress in the use of least-squares principles in connection with finite element applications. The least-squares approach to above problem (1)-(3) results to a minimization problem rather than a saddle point problem. Thus, it provides a simple way to circumvent the inf-sup condition. In particular, for identity matrix \( A = I \), Chang [2] develops a standard least-squares method for solving (1)-(2) with an additional compatibility equation

\[
\nabla \times p := \frac{\partial p_2}{\partial x} - \frac{\partial p_1}{\partial y} = 0 \quad \text{in} \quad \Omega
\]

under two different types of boundary condition, \( p \times \mathbf{n} = 0 \) or \( p \cdot \mathbf{n} = 0 \), where \( \mathbf{n} = (n_1, n_2)^T \) denotes the unit outward normal vector to \( \Gamma \). The method is shown to exhibit optimal rates of convergence both in the \( H^1 \)-norm and \( L^2 \)-norm, and these attractive results
has been extended to three-dimensional case [3].

More recently, Chang [4] proposed a weighted least-squares approach to problem (1)-(3) with an additional compatibility equation similar to (6). This weighted least-squares method also presents optimal rates of convergence both in the $H^1$-norm and $L^2$-norm. Numerical experiments also confirm this assertion. However, the mathematical analysis in [4] for the optimal order of convergence in the $H^1$-norm seems incorrect, because the associated mesh-dependent energy norm is not equivalent to the $H^1$-norm.

In this note, using the technique of the Gauss projection with a suitable inverse assumption, we shall prove that the assertion is still valid. Some remarks are also given. For simplicity, we shall follow the notation in [4]. The definitions of Sobolev spaces on $\Omega$ and $\Gamma$ can be found in, for example, [5].

**II. PROBLEM FORMULATION**

We shall study the weighted least-squares finite element solution to the following first-order system

$$LU := \begin{bmatrix} A\nabla u - p \\ \nabla \cdot p \end{bmatrix} = \begin{bmatrix} 0 \\ -f \end{bmatrix} = -f \quad \text{in } \Omega,$$

$$RU := \begin{bmatrix} 0 & m_1 & m_2 \\ 1 & 0 & 0 \end{bmatrix} U = 0 \quad \text{on } \Gamma,$$

where $U = (u, p_1, p_2)^T$, $m_1 = -(a_{21}n_1 + a_{22}n_2)$, $m_2 = a_{11}n_1 + a_{12}n_2$, and the first boundary condition in (8) follows from $\nabla u \times n = 0$, since $u = 0$ on $\Gamma$. In order to achieve optimal rates of convergence [2], an additional compatibility equation similar to (6) is appended to the system (7)-(8),

$$L_c U := \begin{bmatrix} A\nabla u - p \\ \nabla \cdot p \\ -d \nabla \times A^{-1} p \end{bmatrix} = \begin{bmatrix} 0 \\ -f_c \\ 0 \end{bmatrix} = -f_c \quad \text{in } \Omega,$$

$$R_c U := RU = 0 \quad \text{on } \Gamma,$$

where the subscript $c$ is applied to represent the real system of differential equations and
boundary conditions in our computations, and

\[- d \nabla \times A^{-1} p := a_{22} \frac{\partial p_1}{\partial y} - a_{12} \frac{\partial p_2}{\partial y} + a_{21} \frac{\partial p_1}{\partial x} - a_{11} \frac{\partial p_2}{\partial x} + l_1 p_1 + l_2 p_2 \quad \text{in } \Omega, \quad (11)\]

\[l_1 := d \left( \frac{\partial}{\partial x} \left( \frac{a_{21}}{d} \right) + \frac{\partial}{\partial y} \left( \frac{a_{22}}{d} \right) \right) \quad \text{in } \Omega, \quad (12)\]

\[l_2 := -d \left( \frac{\partial}{\partial x} \left( \frac{a_{11}}{d} \right) + \frac{\partial}{\partial y} \left( \frac{a_{12}}{d} \right) \right) \quad \text{in } \Omega, \quad (13)\]

\[d := \det A \neq 0 \quad \text{on } \overline{\Omega}, \quad (14)\]

The weighted least-squares principle is performed on this enriched first-order system as follows. Let

\[S = \left[ H^1 (\Omega) \right]^3, \quad (15)\]

then define the weighted least-squares quadratic functional \( J_c (V) : S \to \mathbb{R} \) by

\[J_c (V) = \left\| L_c V - \left( -f_c \right) \right\|^2_0 + h^{-1} \left\| R_c V - 0 \right\|^2_0
= \int_\Omega (L_c V + f_c) \cdot (L_c V + f_c) + h^{-1} \int_\Gamma (R_c V) \cdot (R_c V), \quad (16)\]

for all \( V \in S \), where \( h \) is the mesh parameter of the finite element discretization. It is evident that the exact solution \( \overline{U} \in S \) of problem (9)-(10) minimizes (16) since \( J_c (\overline{U}) = 0 \), and a zero minimizer of the functional \( J_c \) on \( S \) solves problem (9)-(10). Thus the weighted least-squares method for (9)-(10) is defined to be the following minimization problem:

\[
\text{Find } \overline{U} \in S \text{ such that } J_c (\overline{U}) = \min_{V \in S} J_c (V). \quad (17)
\]

Taking the first variation,

\[
\frac{d}{d \delta} J_c (\overline{U} + \delta V) |_{\delta = 0} = 0,
\]

we can find that problem (17) is equivalent to

\[
\text{Find } \overline{U} \in S \text{ such that } a_c (\overline{U}, V) = F_c (V) \quad \forall V \in S, \quad (18)
\]
where the bilinear form and the linear form are given, respectively, by
\[
a_c(U, V) := \int_{\Omega} L_c U \cdot L_c V + h^{-1} \int_{\Gamma} R_c U \cdot R_c V \quad \forall V \in S, \tag{19}
\]
\[
F_c(V) := -\int_{\Omega} L_c V \cdot f_c \quad \forall V \in S. \tag{20}
\]
The weighted least-squares finite element scheme is thus defined by
\[
\text{Find } U^h \in S_r^h \text{ such that } a_c(U^h, V^h) = F_c(V^h) \quad \forall V^h \in S_r^h, \tag{21}
\]
where the finite-dimensional subspace \( S_r^h \subset S \) is assumed to satisfy the following approximation assumption: For any \( V \in S \), there exists \( V^h \in S_r^h \) such that
\[
\left\| V - V^h \right\|_m \leq Ch^{s-m} \left\| V \right\|_s, \tag{22}
\]
where \( m = 0, 1, \ m \leq s \leq r \), \( h \) is the mesh parameter, and \( C \) is a positive constant independent of \( h \) and \( V \). Throughout this note, \( C \) will denote a positive constant always independent of \( h \), not necessarily the same in different occurrences.

Let \( e = U - U^h \) denote the exact error, then the weighted least-squares method gives the orthogonality property
\[
a_c(e, V^h) = 0 \quad \forall V^h \in S_r^h, \tag{23}
\]
which implies the best approximation property
\[
a_c(U - U^h, U - U^h) = \inf_{V^h \in S_r^h} a_c(U - V^h, U - V^h). \tag{23}
\]

III. OPTIMAL RATES OF CONVERGENCE

In [4], the following results have been proved.

- For each \( l \in \mathbb{R} \), there exists a positive constant \( \gamma \) such that if \( V \in [H^{l+1}(\Omega)]^3 \), then we have
\[ \| V \|_{l+1} \leq \gamma \left( \| L_c V \|_1 + \| R_c V \|_{l+1/2} \right). \] (24)

- For any \( W \in [H^2(\Omega)]^3 \), there exists a constant \( C > 0 \) independent of \( h \) and \( W \) such that, for \( s = 1, 2 \),

\[ \inf_{W^h \in S_h} \left[ a_c(W - W^h, W - W^h) \right]^{1/2} \leq Ch^{s-1} \| W \|_s. \] (25)

Using (24) with \( l = -1 \), and (25), Chang [4] proves the convergence rate in the \( L^2 \)-norm is optimal,

\[ \| e \|_0 \leq Ch^2 \| U \|_2. \] (26)

However, the proof for the optimal order in the \( H^1 \)-norm is incorrect, cf. [4, page 279], since it is evidently that the mesh-dependent energy norm induced from the inner product \( a_c(\cdot, \cdot) \) is not equivalent to the \( H^1 \)-norm. Here we shall follow Wendland [6] (see also [7], [8], and [9]) to give a new demonstration.

According to (23), (25) with \( s = 1 \), and (4.24) in [4], we get

\[ \| L_c e \|_{s-1} \leq Ch \left[ a_c(e, e) \right]^{1/2} \leq Ch \| U \|_1. \] (27)

Similarly, by (23), (25) with \( s = 1 \), and (4.25) in [4], we have

\[ \| R_c e \|_{s-1/2} \leq Ch \left[ a_c(e, e) \right]^{1/2} \leq Ch \| U \|_1. \] (28)

Combining (24) with \( l = -1 \), (27), and (28), we obtain

\[ \| e \|_0 \leq Ch \| U \|_1. \] (29)

In order to establish the optimal \( H^1 \)-estimates, we need the following Gauss projection [6]:

\[ G^h : S \rightarrow S^h, \quad G^h V \equiv V^h, \]
where \( V^h \) is the solution of the discretized problem (21) corresponding to problem (9)-(10) with suitable data function \( f_c \) such that its unique exact solution is \( V \). Since problem (21) is uniquely solvable (cf. [4], [7], [8], and [9]), the Gauss projection \( G^h \) is well-defined, and we have

\[
G^h V^h = V^h \quad \forall V^h \in S^h_r.
\] (30)

By (29), we obtain

\[
\| G^h U \|_0 = \| U^h \|_0 \leq \| U \|_0 + \| e \|_0 \leq \| U \|_0 + Ch \| U \|_1.
\]

Thus, we can conclude that, for any \( V \in S \),

\[
\| G^h V \|_0 \leq \| V \|_0 + Ch \| V \|_1.
\] (31)

We also need the following inverse assumption on the finite element space \( S^h_r \): There exists a constant \( C > 0 \) independent of \( h \) such that

\[
\| V^h \|_1 \leq C h^{-1} \| V^h \|_0 \quad \forall V^h \in S^h_r.
\] (32)

We are now in the position to prove the optimal order of convergence in the \( H^1 \)-norm. By (30), (32), (31), and (22) for \( V \) replaced by \( U \) with \( m = 0, 1, s = 2 \), we have

\[
\| U - U^h \|_1 \leq \| U - V^h \|_1 + \| U^h - V^h \|_1
\]

\[
= \| U - V^h \|_1 + \| G^h (U - V^h) \|_1
\]

\[
\leq \| U - V^h \|_1 + Ch^{-1} \| G^h (U - V^h) \|_0
\]

\[
\leq \| U - V^h \|_1 + Ch^{-1} \left( \| U - V^h \|_0 + Ch \| U - V^h \|_1 \right)
\]

\[
\leq Ch \| U \|_1.
\] (33)

It is optimal in the \( H^1 \)-norm. Finally, we combine (26) and (33) to obtain

**Theorem 3.1.** There exists a positive constant \( C \) which is independent of \( U \) and \( h \) such that

\[
\| U - U^h \|_0 \leq Ch^2 \| U \|_2.
\] (34)
In addition, if the inverse assumption (32) holds, then we have
\[ \| U - U^h \|_1 \leq Ch^l \| U \|_2. \] (35)

IV. CONCLUDING REMARKS

We conclude this note with the following remarks.

**Remark 4.1.** The inverse assumption (32) on the finite element space \( S^h_r \) is necessary for proving the optimal order of convergence in the \( H^1 \)-norm as above arguments. Inverse assumptions are common in many finite element analyses [10]. More precisely, if the regular family \( \{ \mathcal{T}_h \} \) of triangulations of \( \overline{\Omega} \) associated with the finite element space \( S^h_r \) is quasi-uniform, i.e., there exists a positive constant \( \nu \) independent of \( h \) such that
\[ h \leq \nu \text{diam}(\Omega_i^h), \]
for all \( \Omega_i^h \in \mathcal{T}_h, \mathcal{T}_h \in \{ \mathcal{T}_h \} \), then (32) is satisfied.

**Remark 4.2.** The family of triangulations of \( \overline{\Omega} \) for the numerical experiments in [4] is uniform. Thus, the numerical results demonstrate that the convergence rates in the \( H^1 \)-norm and \( L^2 \)-norm are both optimal.

**Remark 4.3.** The weight \( h^{-1} \) in the least-squares functional (16) is determined by requirements of the supplementary and complementing conditions of [11] or, equivalently, the Petrovski and Lopatinski conditions in [6] which are examined completely in [4].

**Remark 4.4.** This weighted least-squares approach can be extended directly to the two-dimensional grad-div system supplemented with boundary conditions, \( \mathbf{p} \times \mathbf{n} = 0 \) or \( \mathbf{p} \cdot \mathbf{n} = 0 \), on \( \Gamma \) [2], as well as the three-dimensional case [3].

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